MEASURABILITY PROBLEMS IN A METRIZABLE CONVEX COMPACT SET

BY

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ABSTRACT

It is shown that the set of primary points in a metrizable convex compact set is always coanalytic. In particular, it is universally measurable.

In this paper it will be shown that the set of primary points in a metrizable convex compact set is always coanalytic. In particular this set is universally measurable and therefore we obtain a stronger version of Wils fundamental theorem about central desintegration in the metrizable case (see [1], § 8).

Let in the sequel K denote a metrizable convex compact set regularly imbedded in the locally convex real topological vector space E. This means that E can be identified with the dual of the Bancach space $A(K)$ of real continuous affine functions on K ($A(K)$ is equipped with the uniform norm). The set K is located on a hyperplane not containing zero. Usually we consider E with the weak topology. The space E is equipped with the ordering relation defined by the positive cone generated by K (in the sequel we denote this cone by E^+).

The points $x, y \in K$ are strongly disjoint by definition if the convex hull of the faces generated by x and y is again a face. Moreover these faces form a direct convex sum.

Let $M=2B$ where B denotes the unit ball of the Banach space E which, of course, is considered with the usual dual norm. M is a convex compact metrizable set equipped with weak topology. For $x \in K$ let P_x denote the set

$$P_x = \{y \in E \mid -x \leqq y \leqq x\}.$$

Furthermore let R_x denote the set

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$$R_x = \{y \in E \mid 0 \leqq y \leqq x\}.$$

We define

$$S_1 = \{(x,y) \in K^2 \mid P_x \cap P_y = 0\}.$$

From the discussion in [1],§8, it follows that $x, y \in K$ are strongly disjoint if and only if $(x,y) \in S_1$ and moreover $R_x + R_y = 2R_{\frac{1}{2}(x+y)}$.

THEOREM 1. *The set* $\{(x,y) \in K^2 \mid x$ *and* y *are strongly disjoint*$\}$ *is a Borel measurable set of the space* K^2.

PROOF: Let M^* be the space of closed subsets of M equipped with the Effros Borel structure (see [2], [3] and [4] for information about this structure). Because M is compact, there is even a natural compact metrizable topology on M^*.

The graph of P

$$Gr(P) = \{(x,y) \in K \times M \mid -x \leqq y \leqq x\}$$

is closed in $K \times M$. The correspondence $x \to P_x$ is therefore upper semicontinous in the usual sense and hence Effros measurable. A similar argument yields that $x \to R_x$ is Effros measurable. It follows from theorem 3 in the paper [2] that S_1 is Effros measurable (intersection is a measurable operation if and only if the space is a countable union of compact sets). It is easily seen that the mapping $Q(x,y) = R_x + R_y$ from K^2 into M is Effros measurable.

The theorem now follows from the remark stated before, because the set in question is an intersection of two Borel measurable sets. This completes the proof.

THEOREM 2. *The set of primary points in* K *is coanalytic in* K.

PROOF. The point $x \in K$ is primary, by definition, if there does not exist a nontrivial convex combination expressing x as a convex combination of strongly disjoint points. The set of x which allows a decomposition of this type is a continuous image of a Borel set in a compact metric space and hence is analytic. This proves Theorem 2 (note that our argument is analogous to the proof of the fact that the extreme points are a G_δ).

A probability measure u on K is called central if for every Borel subset B such that $\lambda_B = u(B) \in]0,1[$, the barycenters of the probability measures $\lambda_B^{-1} u_B$ and $(1-\lambda_B)^{-1} u_{K/B}$ are strongly disjoint. Let x be a point a of K. Then there exists a unique central probability measure which represents x and is maximal in Choquet's ordering between all central probability measures which represent x. This is Wils' fundamental result (see [1],§8). Wils also proved that $u(C) = 0$

for every compact G_δ set C which does not contain primary points. It follows from Theorem 2 that in the metrizable case, u is concentrated on the set of primary points in the usual measure theoretic sense.

THOEREM 3. *Let $M_1^+(K)$ be the set of probability measures on K equipped with the weak (compact, metrizable) topology. The set of central measures is a coanalytic subset of $M_1^+(K)$.*

PROOF. It is easily seen that the set of all (u,v) in $M_1^+(K)^2$ that is mutually singular is a G_δ, in particular Borel measurable. The barycenters (x_u, x_v) depend measurably on (u,v). Therefore (use Theorem 1) the set of (u,v) which is mutually singular and whose barycenters are not strongly disjoint is Borel measurable in $M_1^+(K)^2$. Now Theorem 3 is proved similarly to Theorem 2.

Some problems in connection with the preceding results remain open. We do not know whether or not Theorem 2 and Theorem 3 are the best possible results. We state the following two conjectures;

If $M_1^+(K)$ is equipped with the Borel structure generated by the weak topology and K is equipped with the Borel structure generated by the analytic sets then the mapping $x \to u_x$ (Wils' unique measure) is measurable.

Any central measure concentrated on the set of primary points is maximal central.

Perhaps the most fascinating problem in the field is that of finding applications analogous to the applications of Choquet theory outside the case of state space of operator algebras.

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